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# TOWARDS OPTIMAL REGULARITY FOR THE FOURTH-ORDER THIN FILM EQUATION IN $\mathbb{R}^N$ : GRAVELEAU-TYPE FOCUSING SELF-SIMILARITY

P. ÁLVAREZ-CAUDEVILLA, J.D. EVANS, AND V.A. GALAKTIONOV

ABSTRACT. An approach to some “optimal” (more precisely, non-improvable) regularity of solutions of the *thin film equation*

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N,$$

where  $n \in (0, 3)$  is a fixed exponent, with smooth compactly supported initial data  $u_0(x)$ , in dimensions  $N \geq 2$  is discussed. Namely, a precise exponent for the Hölder continuity with respect to the spatial radial variable  $|x|$  is obtained by construction of a Graveleau-type focusing self-similar solution. As a consequence, optimal regularity of the gradient  $\nabla u$  in certain  $L^p$  spaces, as well as a Hölder continuity property of solutions with respect to  $x$ , are derived, which cannot be obtained by classic standard methods of integral identities-inequalities.

## 1. INTRODUCTION: TFE–4 AND KNOWN RELATED REGULARITY RESULTS

S1

This paper is devoted to an “optimal regularity” analysis for a fourth-order quasilinear degenerate evolution equation of the parabolic type, called the *thin film equation* (TFE–4), with a fixed exponent  $n > 0$ ,

i1

$$(1.1) \quad u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N,$$

where  $n \in (0, 3)$  is a fixed exponent, and bounded, sufficiently smooth, and compactly supported initial data  $u_0$  (not necessarily positive) with an arbitrary dimension  $N \geq 2$ .

Moreover, we assume  $u_0 = u_0(|x|)$  to be radially symmetric, so solutions  $u = u(|x|, t)$  do the same. These initial conditions could be relaxed (for example  $u_0 \in L^1 \cap L^\infty$ , etc.). However, for simplicity, we have chosen those initial conditions, since the focusing phenomenon to be studied here exists in any reasonable functional class of initial data, including radial ones.

The main result obtained here is an approach to the so-called optimal regularity of solutions of the TFE–4 for the  $N$ -dimensional case with  $N \geq 2$ , ascertaining an “optimal” Hölder continuity exponent in  $x$ .

For the one-dimensional case, these results were obtained in [7], establishing the Hölder continuity in  $C^{0, \frac{1}{2}}$  with respect to the variable  $x$  and in  $C^{0, \frac{1}{8}}$  with respect to  $t$ . However, in the

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$N$ -dimensional case, the optimal regularity has been unsolved besides the interest of the specialized mathematical community; see [8, 10, 16, 17]. In many of those works it is also assumed and proved that the solutions are non-negative for the Cauchy problem (CP). However, in the CP settings, the solutions are oscillatory and sing-changing for all not that large values of  $n > 0$ ; see [2, 13] for recent results and references therein.

One of the difficulties concerning optimal regularity for the equation (1.1) is that it is *not fully divergent*, so the whole “global” regularity theory for the TFE-4 can use just a couple of well-known integral identities/inequalities. Indeed, as discussed in [2] following an integral identities argument as that performed by Bernis–Friedman in 1D fails to show the Hölder continuity in higher dimensions. Consequently, we claim that, for such “partially” (and/or fully non-) divergent equations, a different approach should be put in charge.

**1.1. Methodology: Graveleau-type focusing similarity solutions.** Here, working on the radially symmetric problem of TFE-4, such that

$$u(x, t) = u_*(r, t), \quad \text{with } r = |x|,$$

we base our analysis on the *focusing argument* performed first by Graveleau [14] in a formal sense, and later justified in Aronson–Graveleau [4] for a class of non-negative (via the Maximum Principle, MP) solutions of the *classic porous medium equation* (PME-2):

**pormed** (1.2) 
$$v_t = \Delta(v^m) \quad \text{in } \mathbb{R}^N \times (-\infty, 0), \quad \text{with an exponent } m > 1.$$

Actually, in comparison with (1.1),  $m = 1 + n > 1$ . In particular, the authors in [4] obtained, after constructing a one-parameter family of self-similar solutions, that, when  $N \geq 2$ , there exists  $0 < \mu = \mu(m, N) < 1$  and a radial self-similar solution  $v_*(r, t)$  to the focusing problem with the focusing time  $t = 0^-$  such that<sup>1</sup>

**Ho11** (1.3) 
$$v_*(r, 0^-) = C_* r^\mu, \quad \text{where } C_* \text{ is an arbitrary constant.}$$

This finished a long discussion concerning optimal regularity for the PME-4 after seminar results by Caffarelli–Friedman [9] (see **MathSciNet** for many further extensions) proving, earlier, Hölder continuity of  $v(x, t)$  in a very general setting. The above results of Aronson and Graveleau showed that such a Hölder continuity is, indeed, optimal, and even the corresponding Hölder exponent  $\mu = \mu(m, N)$  from (1.3) can be evaluated, at least, numerically for any  $m > 1$  and  $N \geq 2$ .

Indeed, the blow-up singularity (1.3) shows, actually, not an *optimal* regularity for such PME-2 flows, but the *non-improvable* one, in the sense that for the Hölder exponent of solutions of (1.2)

**Opt1** (1.4) 
$$\text{non-improvable regularity} \quad \Rightarrow \quad \text{Hölder exponent} \not\geq \mu.$$

Fortunately, as we have mentioned above, Caffarelli–Friedman earlier results of the 1980s have been already available at that time, so, together with (1.4) proved that a proper Hölder regularity of any non-negative solutions of the PME-2 is optimal.

These Aronson–Graveleau focusing (singularity–blow-up) ideas for the PME-2 (1.2) and other second-order quasilinear parabolic equations (e.g., for the  $p$ -Laplacian one) got further development and extensions in a number of papers devoted to various important aspects of porous medium flows; see references and results in [3, 5] (see also **MathSciNet**), etc.

More precisely talking about such Graveleau-type similarity approach, from the point of view of applications, in the focusing problem, one assumes that initially there is a non-empty compact

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<sup>1</sup>We do not put further details how to get (1.3), since we will explain this shortly for the TFE-4.

set  $K$  in the complement of the  $\text{supp } v_0$ , where  $v$  vanishes. In other words, there is a hole in the support of the initial value  $v(x, 0) = v_0(x) \geq 0$ , and, in finite time  $T$ , this hole disappears, making the solution  $v$  become positive along the boundary of  $K$  and eventually at all points of  $K$ . Basically due to the finite propagation property that these equations possess (the porous medium equation and the thin film equation). Thus, as the flow evolves the liquid enters  $K$  and eventually reaches all points of  $K$  at the instant  $T$ . We are then interested in how the solution behaves near the focusing time  $T$ .

For the TFE-4 (1.1), there are still any general regularity results even in the radial setting in  $\mathbb{R}^N$ . Therefore, we will follow the lines and the logic of (1.4), i.e., will estimate a certain *non-improvable* Hölder exponent for radial (and, hence, all other) solutions.

Consequently, we use those previous singularity (blow-up) ideas from the porous medium equation (1.2) to establish some properties of the self-similar solutions of the radially symmetric problem of the TFE-4 (1.1) and, eventually, some optimal regularity information about its more general solutions.

Indeed, as we will see through the analysis performed in this paper, to obtain such an “optimal” (non-improvable) regularity for the TFE-4 (1.1) it seems that we must solve the nonlinear focusing problem for that TFE-4.

**1.2. TFE-4 problem settings.** This setting is well-known nowadays (though many things were not fully proved in a general case), so, below, we omit many details; see surveys [16, 17]. Note that, for some values of  $n > 0$  (not that large), focusing similarity solutions to be constructed do not exhibit finite interfaces, so the resulting optimal regularity results are true for any FBP and/or Cauchy problem settings. Principal differences between the CP and the standard FBP settings for the TFE-4 (1.1) are explained in [12, 13], etc.

We recall that the solutions are assumed to satisfy the following zero contact angle boundary conditions:

$$\boxed{\text{i5}} \quad (1.5) \quad \begin{cases} u = 0, & \text{zero-height,} \\ \nabla u = 0, & \text{zero contact angle,} \\ -\mathbf{n} \cdot \nabla(|u|^n \Delta u) = 0, & \text{conservation of mass (zero-flux)} \end{cases}$$

at the singularity surface (interface)  $\Gamma_0[u]$ , which is the lateral boundary of

$$\text{supp } u \subset \mathbb{R}^N \times \mathbb{R}_+, \quad N \geq 1,$$

where  $\mathbf{n}$  stands for the unit outward normal to  $\Gamma_0[u]$ , which is assumed to be sufficiently smooth (the treatment of such hypotheses is not any goal of this paper). For smooth interfaces, the condition on the flux can be read as

$$\lim_{\text{dist}(x, \Gamma_0[u]) \downarrow 0} -\mathbf{n} \cdot \nabla(|u|^n \Delta u) = 0.$$

Next, we denote by

$$\boxed{\text{mass}} \quad (1.6) \quad M(t) := \int u(x, t) \, dx$$

the mass of the solution, where integration is performed over the support. Then, differentiating  $M(t)$  with respect to  $t$  and applying the divergence theorem, we have that

$$J(t) := \frac{dM}{dt} = - \int_{\Gamma_0 \cap \{t\}} \mathbf{n} \cdot \nabla(|u|^n \Delta u) \, d\sigma.$$

The mass is conserved if  $J(t) \equiv 0$ , which is assured by the flux condition in (1.5).

## 2. RADIAL SELF-SIMILAR SOLUTIONS: FOCUSING AND DEFOCUSING CASES

S2

Thanks to the scaling invariant property of these nonlinear parabolic equations, we now construct radially self-similar solutions, i.e., in terms of  $r = |x| > 0$ , with a still unknown value of the parameter  $\alpha > 0$  (clearly, it must be positive, as shown below)

upm

$$(2.1) \quad u_*^\pm(r, t) = (\pm t)^\alpha f(y), \quad y = \frac{r}{(\pm t)^\beta} \text{ for } \pm t > 0, \quad \text{where } \beta = \frac{1+\alpha n}{4} > 0.$$

Here, by the time-translation, we ascribe the blow-up or *focusing* time to  $T = 0^-$ .

We then simultaneously consider two cases:

- (i) **Focusing** (i.e., Graveleau-type) similarity solutions, which play a key role, corresponding to  $(-t)$  in (2.1) and the singular blow-up limit as  $t \rightarrow 0^-$ , and
- (ii) **Defocusing** similarity solutions, with  $(+t)$  in (2.1), playing a secondary role as extensions of the previous ones for  $t > 0$ , i.e., corresponding to the not-that-singular (or, at least, less singular) limit  $t \rightarrow 0^+$ . Actually, these defocusing solutions are well-posed solutions of the Cauchy problem for the TFE-4 for  $t > 0$  with initial data  $u_*(r, 0^-)$  obtained from the previous blow-up limit as  $t \rightarrow 0^-$ .

Substituting (2.1) into (1.1) we arrive at the similarity profiles  $f(y)$  satisfying the following *nonlinear eigenvalue problems*,

eigpm

$$(2.2) \quad \mathbf{B}_n^\pm(\alpha, f) \equiv -\nabla_y \cdot (|f|^n \nabla_y \Delta_y f) \pm \beta y \cdot \nabla_y f \mp \alpha f = 0 \text{ for } y > 0; \quad f'(0) = f'''(0) = 0,$$

where  $\nabla_y$  and  $\Delta_y$  stand for the radial gradient and the radial Laplacian. In (2.2), we present two symmetry conditions at the origin (which should be modified if  $f(y) \equiv 0$  near the origin, i.e., contains a “zero hole” nearby).

To complete these nonlinear eigenvalue settings one needs extra “radiation-type” (or growth-type) conditions at infinity to be introduced next.

Indeed, we actually find radially similarity profiles  $f$  depending on the single variable  $y$ . The operator of the equation (2.2) is then

rad22

$$(2.3) \quad \mathbf{A}_n^\pm(\alpha, f) \equiv -\frac{1}{y^{N-1}} \left[ y^{N-1} |f|^n \left( \frac{1}{y^{N-1}} (y^{N-1} f')' \right)' \right] \pm \beta y f' \mp \alpha f = 0,$$

denoting the radial nonlinear operator as

rad23

$$(2.4) \quad \mathbf{R}_n(\alpha, f) = \frac{1}{y^{N-1}} \left[ y^{N-1} |f|^n \left( \frac{1}{y^{N-1}} (y^{N-1} f')' \right)' \right].$$

Thus, here,  $\alpha > 0$  is a parameter, which, in fact, stands in both cases for admitted real *nonlinear eigenvalues* to be determined, in the focusing Graveleau-type case, via the solvability of the corresponding *nonlinear eigenvalue problem*, accomplished with some special (radiation-type) conditions at infinity, to be introduced shortly.

**Remark 2.1.** In general, with respect to the similarity profiles described above we note that there are two main types of self-similar solutions. For solutions of the first kind the similarity variable  $y$  can be determined *a priori* from dimensional considerations and conservation laws, such as the conservation of mass (1.6) or momentum.

For solutions of the second kind the exponent  $\beta$  (and by relations the exponent  $\alpha$ ) in the similarity variable must be obtained along with the solution by solving a nonlinear eigenvalue problem of the form (2.2).

The first published examples of self-similar solutions of second kind are due to G. Guderley in 1942 [15] studying imploding shock waves, although the term was introduced by Ya.

B. Zel'dovich in 1956 [18]. Further examples might be found in the theory of the collapse of bubbles in compressible fluids or in works on gas motion under an impulsive load; see Barenblatt [6] for an extensive work on this matter.

### 3. MINIMAL GROWTH AT INFINITY (A “NONLINEAR RADIATION CONDITION”) FOR GRAVELEAU-TYPE PROFILES

S3

Here, we consider the blow-up problem (2.3)<sub>−</sub>, i.e., with the lower signs. One concludes that, in order to get a possible discrete set of eigenvalues  $\alpha > 0$ , some extra conditions on the behaviour (in particular, growth) of  $f(y)$  as  $y \rightarrow +\infty$  must be imposed.

Obviously, such a “radiation-type” condition (we use a standard term from dispersion theory) just follows from analyzing all possible types of such a behaviour, which can be admitted by the ODE (2.3)<sub>−</sub>, which is not that a difficult problem. There are two cases:

- (I) *More difficult: there is a “zero hole” for  $f(y)$  near the origin.* Finite interfaces for the TFE-4 are well known in the case of the standard FBP setting and also in the Cauchy one (see [12, 13] and references therein). Then we need to look for profiles  $f(y)$  which vanish at finite  $y = y_0 > 0$  and describe asymptotics of the general solution satisfying zero contact angle and zero flux conditions as  $y \rightarrow y_0^+$

$$f(y) \rightarrow 0, \quad f'(y) \rightarrow 0, \quad -|f|^n f'''(y) \rightarrow 0.$$

This case causes a difficult problem, since the oscillatory behaviour of  $f(y)$  close to the interface is quite tricky for the CP [12], while, for the FBP, it is much better understood.

- (II) *More standard and easier:  $f(0) \neq 0$ .* This case also includes the border possibility  $y_0 = 0$ , i.e., when  $f(0) = 0$ , but  $f \not\equiv 0$  in any arbitrarily small neighbourhood of  $y = 0$ .

**3.1. Minimal and maximal growth at infinity.** This is key for our regularity analysis. Our radial ODE (2.3)<sub>−</sub> admits two kinds of asymptotic behaviour at infinity, i.e., when  $y_0 \rightarrow +\infty$ . For the operator  $\mathbf{A}_n^-(\alpha, f)$ , we state the following result:

theneg

**Proposition 3.1.** *For any  $\alpha > 0$ , the ODE (2.3)<sub>−</sub>, with the operator  $\mathbf{A}_n^-(\alpha, f)$  possesses:*

- (i) *Solutions  $f(y)$  with a minimal growth*

gg1

$$(3.1) \quad f(y) = Cy^\mu(1 + o(1)) \quad \text{as } y \rightarrow +\infty, \quad C \in \mathbb{R},$$

where

param

$$(3.2) \quad \mu = \mu(\alpha, n) = \frac{4\alpha}{1+\alpha n} > 0.$$

- (ii) *Moreover, there exist solutions of (2.3)<sub>−</sub> with a maximal growth*

gg2

$$(3.3) \quad f(y) \sim y^{\mu_0}, \quad \text{as } y \rightarrow +\infty, \quad \text{where } \mu_0 = \frac{4}{n}.$$

**Remark.**

- It is important to mention that the expression (3.3) for solutions of the maximal growth does not include the “additional or extra” corresponding *oscillatory component*  $\varphi(s)$ , with  $s = \ln y$ , and shows just the growth behaviour of its “envelope” with algebraic growth. Such an oscillatory maximal behaviour will be introduced and studied later.

- Note also that, as  $n \rightarrow 0^+$ , we have that

$$\boxed{\text{gg81}} \quad (3.4) \quad \frac{4}{n} \rightarrow +\infty,$$

which corresponds to an exponential oscillatory growth for the linear problem occurring for  $n = 0$ ; see next section.

- The difference between (3.1) and (3.3), which implies such terms as minimal and maximal growth at infinity, is obvious:

$$\boxed{\text{gg3}} \quad (3.5) \quad \text{for any } \alpha > 0, \quad \frac{4}{n} > \frac{\alpha}{\beta} = \frac{4\alpha}{1+\alpha n}.$$

*Proof.* This follows from a balancing of linear and the nonlinear operators in this ODE, though a rigorous justification is rather involved and technical. A formal derivation is surely standard and easy:

(i) In this first case, we assume a linear asymptotic as  $y \rightarrow +\infty$  (assuming simple radial behaviour  $f \sim y^\mu$ )

$$\boxed{\text{kk3}} \quad (3.6) \quad \frac{1+\alpha n}{4} y f' - \alpha f + \dots = 0 \implies f(y) \sim y^\mu, \quad \text{where } \mu = \frac{4\alpha}{1+\alpha n} > 0.$$

Since this behaviour is asymptotically linear, eventually, we get an arbitrary constant  $C \neq 0$  in (3.1). A full justification of existence of such orbits is straightforward, since the nonlinear term in the ODE is then negligible. So that in the equivalent integral equation though a singular term, it produces a negligible perturbation.

(ii) On the other hand, the solutions are also bounded by a maximal growth, which in this case comes from non-linear asymptotics that balance all three operators

$$\boxed{\text{kk2}} \quad (3.7) \quad \mathbf{R}_n(f) + \frac{1+\alpha n}{4} y f' - \alpha f = 0 \implies f(y) \sim y^{\frac{4}{n}} \quad \text{as } y \rightarrow +\infty,$$

where we again indicate the envelope behaviour of this oscillatory bundle. A justification here, via Banach's contraction principle is even easier, but rather technical, so we omit details.  $\square$

Overall this allows us to formulate such a condition at infinity, which now takes a clear “minimal nature” such that solutions  $f$  are now bounded at infinity by a function

$$\boxed{\text{mingro}} \quad (3.8) \quad f(y) = C y^\mu (1 + o(1)), \quad \text{with } \mu = \frac{4\alpha}{1+\alpha n} > 0.$$

Obviously, thus we just need a global solution  $f(y)$  of our ODE (2.3)<sub>-</sub> in  $\mathbb{R}_+$ , satisfying the minimal growth (3.1). Indeed, for such profiles, there exists a finite limit at the focusing (blow-up) point:

$$\boxed{\text{gg5}} \quad (3.9) \quad u_*(r, t) \rightarrow C r^\mu \quad \text{as } t \rightarrow 0^-, \quad \text{with } \mu = \frac{\alpha}{\beta},$$

uniformly on compact intervals in  $r = |x| \geq 0$ . One can see that, for any maximal profile as in (3.3), the limit as in (3.9) is infinite, so that such similarity solutions do not leave a finite trace as  $t \rightarrow 0^-$ .

However, the above proposition leaves aside the principle question on the dimensions of the corresponding minimal and maximal bundles as  $y \rightarrow +\infty$ , which is not a straightforward problem. Note that the latter is actually supposed to determine the strategy of a well posed shooting of possible solutions of the above focusing problem.

And the main problem is how to find those admitted values of nonlinear eigenvalues  $\{\alpha_k\}_{k \geq 1}$  (possibly and hopefully, a discrete set), for which  $f = f_k(y)$  exist producing finite limits as in (3.9).

To clarify those issues, we consider the much simpler linear case when  $n = 0$  and, subsequently, pass to the limit as  $n \rightarrow 0^+$  in (2.3). This analysis will provide us, eventually, with some qualitative information about the solutions, at least when  $n$  is very close to zero.

#### 4. THE LINEAR PROBLEM: DISCRETIZATION OF $\alpha$ FOR $n = 0$

**S.lin**

For  $n = 0$ , the TFE-4 (1.1) becomes the classic *bi-harmonic equation*

**bi.1**

$$(4.1) \quad u_t = -\Delta^2 u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_-.$$

Of course, solutions of (4.1) are analytic in both  $x$  and  $t$ , so any focusing for it makes no sense. However, we will show that Graveleau-type “focusing solutions” for (4.1) are rather helpful to predict some properties of true blow-up self-similar solutions of (1.1), at least, for small  $n > 0$ ; see below.

**4.1. Maximal and minimal bundles.** Thus, we consider the same “focusing” solutions (2.1)– for (4.1), that take a simpler form

**b1**

$$(4.2) \quad u_*(r, t) = (-t)^\alpha f(y), \quad y = \frac{r}{(-t)^{1/4}} \quad (\beta = \frac{1}{4}).$$

Then, the corresponding linear radial ODE (2.3) takes also a simpler form

**b2**

$$(4.3) \quad \mathbf{A}_{0,y}^-(\alpha, f) \equiv -\frac{1}{y^{N-1}} [y^{N-1} (\frac{1}{y^{N-1}} (y^{N-1} f')')]' - \frac{1}{4} y f' + \alpha f = 0.$$

We first calculate the solutions of (4.3) with a maximal behaviour: those are exponentially growing solutions,

**b3**

$$(4.4) \quad f(y) \sim e^{ay^\gamma} \quad \text{as} \quad y \rightarrow +\infty \quad \implies \quad a^3 = -\frac{1}{4} \left(\frac{3}{4}\right)^3.$$

This characteristic equation gives two roots with  $\text{Re}(\cdot) > 0$ :

**b4**

$$(4.5) \quad a_{1,2} = \frac{3}{4} 4^{-\frac{1}{3}} \left[ \frac{1}{2} \pm i \frac{\sqrt{3}}{2} \right] \equiv c_0 \pm i c_1.$$

and one negative root (this actually goes to the bundle of minimal solutions; see below)

**b5**

$$(4.6) \quad a_3 = -\frac{3}{4} 4^{-\frac{1}{3}} < 0.$$

Hence, the bundle of maximal solutions is oscillatory as  $y \rightarrow +\infty$  (including the multiplicative algebraic factor):

**b51**

$$(4.7) \quad f(y) \sim y^{-\frac{2}{3}(N+2\alpha)} e^{c_0 y^{4/3}} [C_1 \cos(c_1 y^{\frac{4}{3}}) + C_2 \sin(c_1 y^{\frac{4}{3}})] + \dots, \quad C_{1,2} \in \mathbb{R}.$$

In fact, obviously, this bundle is 3D since it includes the 1D sub-bundle of exponentially decaying solutions with the exponent (4.6). However, in a shooting procedure below, we intend to get rid of just two coefficients in (4.7):  $C_1 = C_2 = 0$ .

The minimal behaviour (3.1) now reads

**b6**

$$(4.8) \quad f(y) = C y^{4\alpha} (1 + o(1)), \quad C \in \mathbb{R} \quad \left( \frac{\alpha}{\beta} = 4\alpha \right).$$

The whole bundle of such minimal solutions is 2D: besides the parameter  $C$  in (4.8) it includes a 1D sub-bundle of exponentially decaying solutions with the exponent (4.6), so that, overall, the minimal solutions compose a 2D family:

**b7**

$$(4.9) \quad f(y) \sim C y^{4\alpha} (1 + o(1)) + D e^{a_3 y^{4/3}} (1 + o(1)), \quad C, D \in \mathbb{R}.$$

Justification of both behaviours is, indeed, simpler for such a standard linear ODE problem.



Finally, passing to the limit as in (3.9) then gives, for minimal solutions, a finite “focusing trace”:

**b8** (4.10) 
$$u_*(r, t) \rightarrow Cr^{4\alpha} \quad \text{as } t \rightarrow 0^-.$$

We need to explain why those trivial results are so important in what follows. The conclusion from (4.10) is straightforward: since  $u_*(r, 0^-)$  must be analytic then we have the following.

**Pr.anal** **Proposition 4.1.** *For the above linear “focusing eigenvalue problem” (4.3), (4.9), there exists not more than a countable set of admissible eigenvalues  $\{\alpha_k\}_{k \geq 1}$  given by*

**b9** (4.11) 
$$\alpha_k = \frac{k}{2}, \quad k = 1, 2, 3, \dots$$

*Proof.* The function  $r^{4\alpha}$  is analytic at  $r = 0$  only for values  $\alpha_k$  from (4.11), i.e.,  $4\alpha$  must be a real even integer, say  $2k$ . Note also that, in the present problem, all functions are analytic, so that, automatically, the set of roots  $\{\alpha_k\}$  is discrete with a possible accumulation point at infinity only.  $\square$

**Remark.** Solving the *linear eigenvalue problem*:

$$\mathbf{A}_{0,y}^-(\alpha_k, f_k) = 0 \quad \text{in } \mathbb{R}, \quad f_k \in L_\rho^2(\mathbb{R}),$$

then, it seems that the nonlinear eigenvalue problem (2.3)<sub>-</sub> formally reduces to the classic linear eigenvalue problem (4.3) at  $n = 0$ , another reason to call (2.3) a *nonlinear eigenvalue problem*.

Also, the values of the parameter  $\alpha$  can be written as shifting from the eigenvalues of the eigenvalue problem

$$\mathbf{B}^* f \equiv -\frac{1}{y^{N-1}} \left[ y^{N-1} \left( \frac{1}{y^{N-1}} (y^{N-1} f')' \right)' \right] - \frac{1}{4} y f' = \lambda f,$$

analysed in [11] and whose discrete spectrum takes the form

$$\sigma(\mathbf{B}^*) = \{\lambda_k = -\frac{k}{4}; k = 0, 1, \dots\}.$$

Hence,

$$\alpha_k = \lambda_k - \frac{k+1}{4}, \quad \text{with } k = 1, 2, 3, \dots,$$

having a countable family of eigenvalues for the problem (4.3).

Moreover, note that  $\mathbf{A}_{0,y}^-(\alpha, f)$  (4.3) is a non-symmetric linear operator, which is bounded from  $H_\rho^4(\mathbb{R})$  to  $L_\rho^2(\mathbb{R})$  with the exponential weight given by (4.7), i.e.

$$\rho(y) = e^{-c_0|y|^{4/3}}, \quad c_0 > 0 \text{ small and defined by (4.5).}$$

To PAC: your business again!!!! What weighted space, a crucial problem...
---

to VAG: is that correct??
---------------------------

To PAC: seems OK, no problem, done...
---------------------------------------

**4.2. Well-posed shooting procedure.** We now discuss a practical procedure to obtain the linear eigenvalues  $\alpha_k$ . We perform standard shooting from  $y = 0$  to  $y = +\infty$  for the ODE (4.3) by posing four conditions at the origin:

either

**sh1** (4.12) 
$$f(0) = 1 \text{ (normalisation), } f''(0) = 0, \quad f'(0) = f'''(0) = 0 \text{ (symmetry).}$$

or

**sh2** (4.13) 
$$f(0) = 0, \quad f''(0) = 1 \text{ (normalisation), } f'(0) = f'''(0) = 0 \text{ (symmetry).}$$

These two sets of conditions correspond to partitioning of the eigenfunctions into two subsets with the corresponding properties at the origin of either  $f(0) = 1, f''(0) = 0$  or  $f(0) = 0, f''(0) = 1$  together with symmetry conditions. Explicitly, the first four eigenfunctions are

$$\begin{aligned} k = 1 : \quad \alpha_1 &= \frac{1}{2}, \quad f_1(y) = \frac{1}{2}y^2, \\ k = 2 : \quad \alpha_2 &= 1, \quad f_2(y) = 1 + \frac{1}{8N(N+2)}y^4, \\ k = 3 : \quad \alpha_3 &= \frac{3}{2}, \quad f_3(y) = \frac{1}{2}y^2 + \frac{1}{48(N+2)(N+4)}y^6, \\ k = 4 : \quad \alpha_4 &= 2, \quad f_4(y) = 1 + \frac{1}{4N(N+2)}y^4 + \frac{1}{192N(N+2)(N+4)(N+6)}y^8, \end{aligned}$$

where the above properties are immediately apparent.

Numerically we may illustrate the appearance of the eigenfunctions and their eigenvalues through consideration of an Initial Value Problem (IVP). The linear ODE (4.3) is solved numerically using ode15s (with tight error tolerances AbsTol=RelTol= $10^{-13}$ ) and subject to either (4.12) or (4.13) as initial conditions. The far-field behaviour (4.7) is extracted from the numerical solution. The oscillatory component in the far-field behaviour is revealed by considering the scaled solution

scaledf

$$(4.14) \quad f(y)y^{\frac{2}{3}(N+2\alpha)}e^{-c_0y^{4/3}}.$$

A least squares fitting of this function to the remaining oscillatory component in (4.7) over a suitable interval for large  $y$ , allows the determination of the constants  $C_{1,2}$ . The large  $y$  interval taken was typically [250,300]. The scaled function (4.14) is shown in Figure 1 for each of the two types of initial conditions in the  $N = 1, 2$  and 3 cases. Two selected values of the parameter  $\alpha$  are taken in each case. The figures illustrate convergence to the oscillatory part of the far-field behaviour, with the extracted least squares estimates of the constants  $C_{1,2}$  shown inset for each of the two *alpha* values. Figure 2 shows the variation of the far-field constants  $C_{1,2}$  with  $\alpha$  in the  $N = 1, 2, 3$  cases and the two initial conditions. The constants  $C_{1,2}$  both vanish precisely at the eigenvalues, when  $\alpha = \alpha_k$ , the first five being shown in Figure 2 in each  $N$  case. The magnitude of the constants  $C_{1,2}$  grow rapidly as  $\alpha$  increases, making determination of further eigenvalues more difficult.

This approach demonstrates the validity of numerical determination of the eigenvalues and eigenfunctions by choosing  $\alpha$  to minimise the far-field behaviour (4.7).

**4.3. A “homotopic” transition to small  $n > 0$ : some key issues.** Next, we are going to use the above linear results to predict true nonlinear eigenvalues  $\alpha_k(n)$  for the TFE-4, at least, for sufficiently small  $n > 0$ . By continuity (to be discussed later on), we now know that

b10

$$(4.15) \quad \alpha_k(0) = \frac{k}{2}, \quad k = 1, 2, 3, \dots$$

Another important conclusion: since, for  $n = 0$ , any  $f_k(0) \neq 0$  (this could happen only accidentally, with a probability 0), we can also expect that

b11

$$(4.16) \quad \text{for small } n > 0, \quad f_k(0) \neq 0,$$

meaning that, in this case, we do not need to perform a shooting from the interface point, with a quite tricky behaviour nearby.

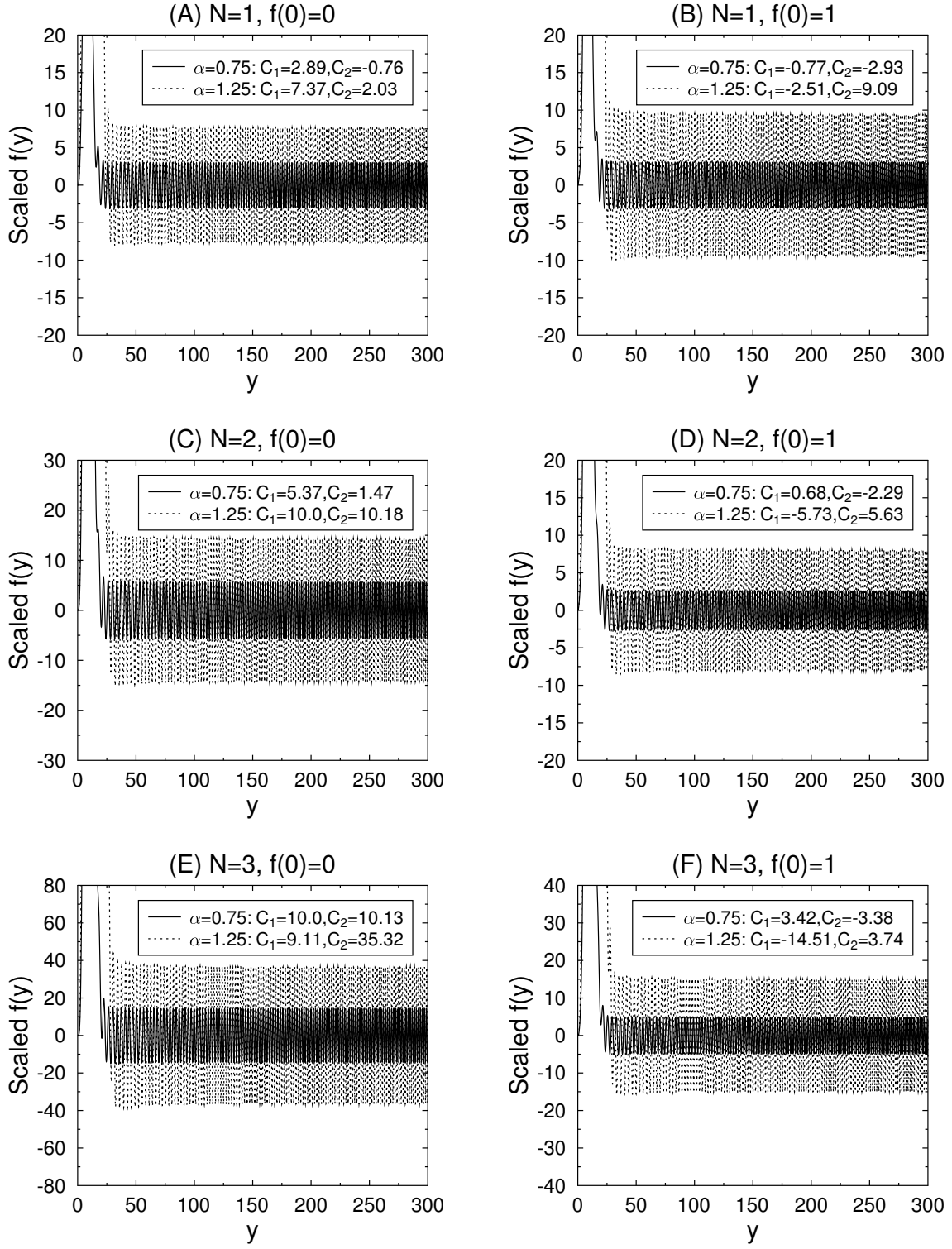


FIGURE 1. Numerical illustration of the oscillatory component in the far-field behaviour (4.7) in the linear case. In each dimensional case  $N = 1, 2, 3$  scaled profiles (4.14) are shown for two selected values of the parameter  $\alpha$  for each initial condition (4.12) and (4.13). The extracted least squares values of the far-field constants  $C_{1,2}$  are stated inset in each figure.

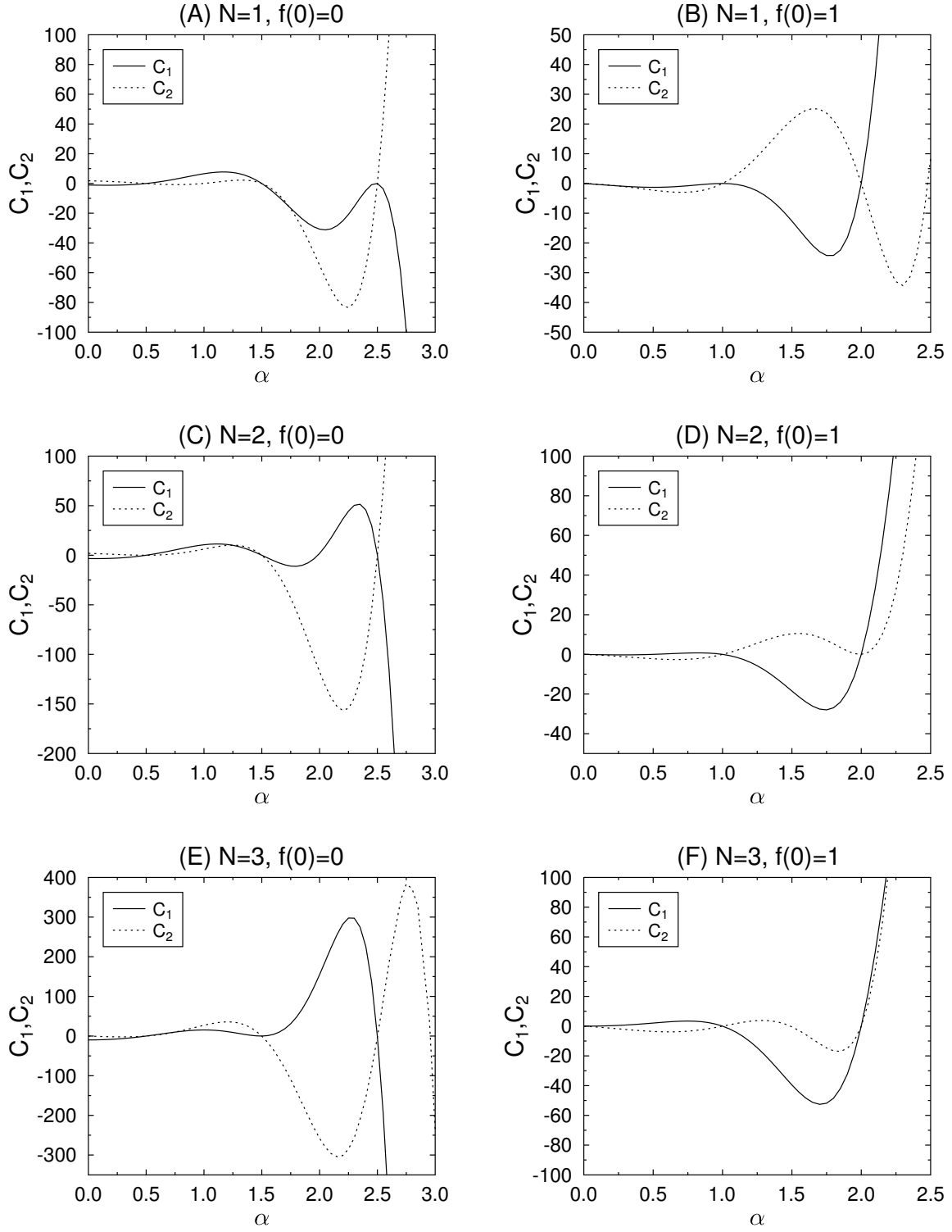


FIGURE 2. Numerical determination of the far-field constants  $C_{1,2}$  for (4.7) in the stated dimensional and initial condition cases. Shown are their variation with the parameter  $\alpha$ . The coincident zeros correspond to the vanishing of the far-field maximal bundle yielding the eigenvalues of the linear problem  $\alpha = \alpha_k$ .

Also, we conclude from (4.7) that:

$$\boxed{\text{b12}} \quad (4.17) \quad \text{for small } n > 0, \text{ maximal bundle is 2D and oscillatory as } y \rightarrow +\infty.$$

Those properties will aid progress on the nonlinear eigenvalue problem.

**Passing to the limit  $n \rightarrow 0^+$  in the nonlinear eigenvalue problem.** Subsequently, we perform a homotopy deformation from the self-similar equation (2.3)<sub>-</sub> (with lower signs) of the TFE-4 (1.1) to the linear radial ODE (4.3) corresponding to the *classical bi-harmonic parabolic equation* (4.1). In particular we construct a continuous deformation from the radial equation (2.3)<sub>-</sub> to the linear equation (4.3) for which we know solutions explicitly.

It is clear that the CP for the *bi-harmonic equation* (4.1) is well-posed and has a unique solution given by the convolution

$$\boxed{\text{b.11}} \quad (4.18) \quad u(x, t) = b(x - \cdot, t) * u_0(\cdot),$$

where  $b(x, t)$  is the fundamental solution of the operator  $D_t + \Delta^2$ .

Due to our analysis is possible to establish a connection between the radial solutions of (1.1) and the corresponding to (4.1) via the self-similar associated equations when  $n \rightarrow 0^+$ . To this end we apply the Lyapunov–Schmidt method to ascertain qualitative properties of the self-similar equation (2.3)<sub>-</sub> following a similar analysis as the one carried out in [1].

Thus, as we already know, the operator  $\mathbf{A}_{0,y}^-(\alpha, f)$  defined by (4.3) produces a countable family of eigenvalues

$$\alpha_k \equiv \alpha_k(0) = \frac{k}{2}, \quad \text{with } k = 0, 1, \dots$$

Note also that, (4.3) admits a complete and closed set of eigenfunctions being *generalized Hermite polynomials*, which exhibit finite oscillatory properties.

This oscillatory issue seems to be crucial. In fact, in [12] was observe that a similar analysis of blow-up patterns for a TFE-4 like (1.1) did not detect any stable oscillatory behaviour of solutions near the interfaces of the radially symmetric associated equation. All the blow-up patterns turned out to be nonnegative, which is a specific feature of the PDE under consideration therein. However, this does not mean that blow-up similarity solutions of the CP do not change sign near the interfaces or inside the support. Actually, it was pointed out that local sign-preserving property could be attributed only to the blow-up ODE and not to the whole PDE (1.1). Hence, the possibility of having oscillatory solutions cannot be ruled out for every case.

**4.4. Branching/bifurcation analysis.** Now, we construct a continuous deformation such that the patterns occurring for the nonlinear eigenvalue problem (2.3)<sub>-</sub> are homotopically connected to the ones of the equation (4.3). Thus, we assume for small  $n > 0$  in (2.3) the following expansions:

$$\boxed{\text{br3}} \quad (4.19) \quad \alpha_k(n) := \alpha_k + \mu_{1,k}n + o(n), \quad |f|^n \equiv |f|^n = e^{n \ln |f|} := 1 + n \ln |f| + o(n),$$

where the last one is assumed to be understood in a weak sense. The second expansion cannot be interpreted pointwise for oscillatory changing sign solutions  $f(y)$ , though now these functions are assumed to have *finite* number of zero surfaces (as the generalized Hermite polynomials for  $n = 0$  do). Indeed, as discussed in [1] for (2.3) this is true if the zeros are transversal.

Furthermore, in order to apply the Lyapunov–Schmidt Branching analysis we suppose the expansion

$$\boxed{\text{br12}} \quad (4.20) \quad f = \sum_{|\beta|=k} c_\beta f_\beta + V_k, \quad \text{for every } k \geq 1,$$

under the natural “normalizing” constraint

$$\boxed{\text{nor}} \quad (4.21) \quad \sum_{|\beta|=k} c_\beta = 1.$$

Moreover, we write

$$\{f_\beta\}_{|\beta|=k} = \{f_1, \dots, f_{M_k}\},$$

as the natural basis of the  $M_k$ -dimensional eigenspace, with  $M_k \geq 1$ , such that

$$f_k = \sum_{|\beta|=k} c_\beta f_\beta \quad \text{and} \quad V_k = \sum_{|\beta|>k} c_\beta f_\beta,$$

with  $V_k \in Y_k$  where  $Y_k$  is the complementary invariant subspace of corresponding kernel. In particular, we consider the expansion

$$V_k := n\Phi_{1,k} + o(n).$$

Thus, applying the Fredholm alternative, we obtain the existence of a number of branches emanating from the solutions  $(\alpha_k, f_k)$  at the value of the parameter  $n = 0$ . The number of branches will depend on the dimension of the eigenspace; see [1] for any further comments and a detailed analysis of a similar branching analysis.

The previous discussion can be summarized in the following Lemma.

**Lemma 4.1.** *The patterns occurring for equation (4.3), i.e. the ones for (2.3) for  $n = 0$ , act as branching points of nonlinear eigenfunctions of the Cauchy problem for the operator (2.3)<sub>-</sub>, at least when the parameter  $n$  is sufficiently close to zero.*

**Remark.** It turns out that using classical branching theory “nonlinear eigenfunctions”  $f(y)$  of changing sign, which satisfies the *nonlinear eigenvalue problem* (2.3)<sub>-</sub> (with an extra “radiation-minimal-like” condition at infinity at least, for sufficiently small  $n > 0$ , can be connected with eigenfunctions  $f_k$  of the linear problem (4.3).

## 5. NON-IMPROVABLE REGULARITY FOR THE TFE-4

The following result is a straightforward consequence of our focusing self-similar analysis.

$\boxed{\text{theneg2}}$  **Theorem 5.1.** *Let, for a fixed  $n > 0$ , the nonlinear eigenvalue problem (2.3)<sub>-</sub>, (3.1) have a nontrivial solution (eigenfunction)  $f_k(y)$  for some eigenvalue  $\alpha_k > 0$ , i.e., there exists a self-similar focusing solution (2.2)<sub>-</sub> exhibiting the finite-time trace (3.9). Then:*

(i) *For the general Cauchy problem for the TFE-4 (1.1), even in the radial setting, the Hölder continuity exponent of solutions cannot exceed<sup>2</sup>*

$$\boxed{\text{h11}} \quad (5.1) \quad \mu_k(n, N) = \frac{\alpha_k}{\beta_k} \equiv \frac{4\alpha_k}{1+n\alpha_k}.$$

(ii) *Let there exist  $\mu_k < 1$ , i.e.,  $\alpha_k < \frac{1}{4-n}$ , for  $n \in (0, 2)^3$ . Then, for the TFE-4 (1.1),*

$$\boxed{\text{h12}} \quad (5.2) \quad \nabla u_*(r, 0^-) \in L_{\text{loc}}^p(\mathbb{R}^N) \quad \text{iff} \quad p \in [1, p_*), \quad p_*(n, N) = \frac{N}{1-\mu_k},$$

*so that, for (even radial) solutions of (1.1), in general, for any  $t > 0$ ,*

$$\boxed{\text{h13}} \quad (5.3) \quad \nabla u(x, t) \notin L_{\text{loc}}^{p_*}(\mathbb{R}^N).$$

<sup>2</sup>We mean  $C^{l+\sigma}$ , if  $\mu_k \geq 1$ , that, not that surprisingly, happens for all small  $n > 0$ .

<sup>3</sup>Actually, for smaller  $n$ ; for all large  $n$ 's solutions of the TFE-4 are known to be strictly positive, [7]).

**Remark.** Basically  $|\nabla u_*(r, 0^-)| \in L^p_{\text{loc}}(\mathbb{R}^N)$  if and only if

$$\int_{\mathbb{R}^N} |\nabla u_*(r, 0^-)|^p = C \int_0^1 r^{N-1} (r^{\mu_k-1})^p < \infty.$$

Thus, since  $0 < \mu_* < 1$  we find that to have that integral bounded we need that, evaluating the coefficients in the second integral

$$p < p_* \equiv p_*(n, N) = \frac{N}{1-\mu_k(n, N)}.$$

**Remark.** Of course, it can happen that  $\mu_k > 1$ , so focusing does not supply us with a truly Hölder continuous focusing trace but rather than a  $C^{l+\sigma}$ -trace. Actually, by continuity in  $n$ , exactly this happens for  $n > 0$  small, where  $\mu_1(0, N) = 2$ .

We expect that the minimal value of  $\mu_k$  in (5.1) is attained at  $k = 1$ , but cannot prove this for all  $n > 0$  (for small ones, this is obvious). Note again that, for all large  $n \geq 2$ , such a focusing is not possible in principle, since the solutions must remain strictly positive for all times, [7].

Thus, this optimal (non-improvable) regularity results for the TFE-4 (and, seems for many other parabolic equations) depends on the solvability of the nonlinear eigenfunction focusing problem.

## 6. OSCILLATORY STRUCTURE OF MAXIMAL SOLUTIONS FOR $n > 0$

S.maxosc

Let us now consider  $n \in (0, 2)$ . For the ODE (2.3), we will try the same *ansatz* as in [13]. Namely, we will find solutions of the maximal type, with the envelope as in (3.3). We introduce a corresponding *oscillatory* component as follows:

$$(6.1) \quad f(y) = y^\mu \varphi(s), \quad s = \ln y, \quad \mu = \frac{4}{n} \quad \text{for } y \gg 1.$$

Then,

$$\begin{aligned} f' &= y^{\mu-1}(\dot{\varphi} + \mu\varphi), \quad f'' = y^{\mu-2}[\ddot{\varphi} + (\mu-1)\dot{\varphi} + \mu(\mu-1)\varphi], \\ f''' &= y^{\mu-3}[\ddot{\ddot{\varphi}} + 3(\mu-1)\ddot{\varphi} + (3\mu^2 - 6\mu + 2)\dot{\varphi} + \mu(\mu^2 - 3\mu + 2)\varphi], \end{aligned}$$

where  $' = d/dy$  and  $\dot{\phantom{x}} = d/ds$ . Substituting these expressions into (2.3) yields the following fourth-order homogeneous ODE for  $\varphi(s)$

phieqn

$$(6.2) \quad \begin{aligned} &\ddot{\ddot{\varphi}} + 2(N-4+2\mu)\ddot{\ddot{\varphi}} + (6\mu^2 + 6(N-4)\mu + 11 + (N-1)(N-9))\ddot{\varphi} \\ &+ 2(2\mu + N-4)(\mu^2 + (N-4)\mu + 2 - N)\dot{\varphi} + \mu(\mu-2)(\mu^2 + 2(N-3)\mu + 3 + (N-1)(N-5))\varphi \\ &+ n\left(\frac{\dot{\varphi}}{\varphi} + \mu\right)\left[\ddot{\ddot{\varphi}} + (N-4+3\mu)\ddot{\ddot{\varphi}} + (3\mu^2 + 2(N-4)\mu + 4 - 2N)\dot{\varphi} + \mu(\mu-2)(N-2+\mu)\varphi\right] \\ &+ \left(\frac{1}{4}(1+n\alpha)(\dot{\varphi} + \mu\varphi) - \alpha\varphi\right)|\varphi|^{-n} = 0. \end{aligned}$$

We mention that this equation is autonomous and thus may be reduced to third-order, although we do not utilize this reduction here. Figure 3 illustrates the periodic nature of the solutions for  $\phi$ , at least for  $n$  small enough. Since the oscillations occur over such a large range, we use the following transformation to allow the oscillations to be visible on the plots,

tphi

$$(6.3) \quad t(\phi(s)) = \begin{cases} \ln \phi(s) + 1, & \text{if } \phi(s) > 1, \\ \phi(s), & \text{if } -1 < \phi(s) < 1, \\ -\ln(-\phi(s)) - 1, & \text{if } \phi(s) < -1. \end{cases}$$

Plotted are numerical solutions for  $\phi$  in the two cases  $\alpha = 0.5, 1$  with  $N = 1$ . The subplots illustrate the change in the profile behaviour with  $n$ . The numerics support the proposition that the family (6.1) is 2D, composed of:

- (i) a 1D stable manifold  $\varphi_*(s)$ ,  
and
- (ii) a phase shift  $\varphi_*(s + s_0)$  for any  $s_0 \in \mathbb{R}$ .

As we have seen earlier, these two properties are true for  $n = 0$ , and, hence, by continuity, we conjecture remain true for small  $n > 0$ . However, the periodic exponential structure, with the linear ODE for  $n = 0$  is replaced by a more difficult one (6.1) for  $n > 0$ . Nevertheless, the periodic nature of such a behaviour appears universal, and is expected to remain for larger  $n$ . The numerics though do suggest that it is lost (in a homoclinic-heteroclinic bifurcation) for  $n$  near to 0.8 or 0.9.

To reconcile with the known behaviour in the linear case, we now study the behaviour of the periodic solutions for small  $n > 0$ . To reveal the limiting oscillatory behaviour as  $n \rightarrow 0$ , we keep the leading terms of the coefficients in (6.2) as  $\mu \rightarrow \infty$  to obtain

$$\boxed{\text{phismalln}} \quad (6.4) \quad \ddot{\varphi} + 4\mu\ddot{\varphi} + 6\mu^2\ddot{\varphi} + 4\mu^3\dot{\varphi} + \mu^4\varphi + \frac{1}{4}(\dot{\varphi} + \mu\varphi)|\varphi|^{-n} = 0.$$

We now rescale as follows

$$\boxed{\text{AAA.1}} \quad (6.5) \quad \varphi(s) = \mu^{-\frac{3}{n}}\hat{\varphi}(\hat{s}), \quad s = \frac{\hat{s}}{\mu},$$

leading to

$$\boxed{\text{phihat}} \quad (6.6) \quad \ddot{\hat{\varphi}} + 4\dot{\hat{\varphi}} + 6\ddot{\hat{\varphi}} + 4\dot{\hat{\varphi}} + \hat{\varphi} + \frac{1}{4}(\dot{\hat{\varphi}} + \hat{\varphi})|\hat{\varphi}|^{-n} = 0,$$

where here  $\dot{\phantom{x}}$  denotes  $d/d\hat{s}$ .

For  $n = 0$ , the equation (6.6) becomes linear,

$$\boxed{\text{phihatn0}} \quad (6.7) \quad \ddot{\hat{\varphi}} + 4\ddot{\hat{\varphi}} + 6\ddot{\hat{\varphi}} + 4\dot{\hat{\varphi}} + \hat{\varphi} + \frac{1}{4}(\dot{\hat{\varphi}} + \hat{\varphi}) = 0,$$

with the characteristic equation  $(m+1)((m+1)^3 + \frac{1}{4}) = 0$  for exponential solutions  $\hat{\varphi}(\hat{s}) = e^{m\hat{s}}$ . The roots of the characteristic equation are linked to those of the controlling factor in (4.4) being  $m+1 = \frac{4}{3}a_1, \frac{4}{3}a_2, \frac{4}{3}a_3$  and 0. Consequently we obtain the dominant asymptotic behaviour

$$\boxed{\text{phihatasy}} \quad (6.8) \quad \hat{\varphi}(\hat{s}) \sim e^{(-1+\frac{4}{3}c_0)\hat{s}} \left[ \hat{A}_1 \cos\left(\frac{4}{3}c_1\hat{s}\right) + \hat{A}_2 \cos\left(\frac{4}{3}c_1\hat{s}\right) \right] \quad \text{as } \hat{s} \rightarrow \infty,$$

for arbitrary constants  $\hat{A}_{1,2}$ . Denoting the independent variable by  $\hat{y}$  rather than  $y$  for convenience, we thus have the small  $n$  behaviour

$$(6.9) \quad f(\hat{y}) \sim \hat{y}^{\frac{4}{n}} \left(\frac{4}{n}\right)^{-\frac{3}{n}} \hat{\varphi}\left(\frac{4}{n} \ln(\hat{y})\right).$$

This may be reconciled with the expression in (4.7) through the identifications

$$\hat{y} = e^{\frac{3n}{16}y^{4/3}}, \quad \hat{A}_{1,2} = \left(\frac{4}{n}\right)^{\frac{3}{n}} C_{1,2}.$$



$N=1, \alpha=0.5$

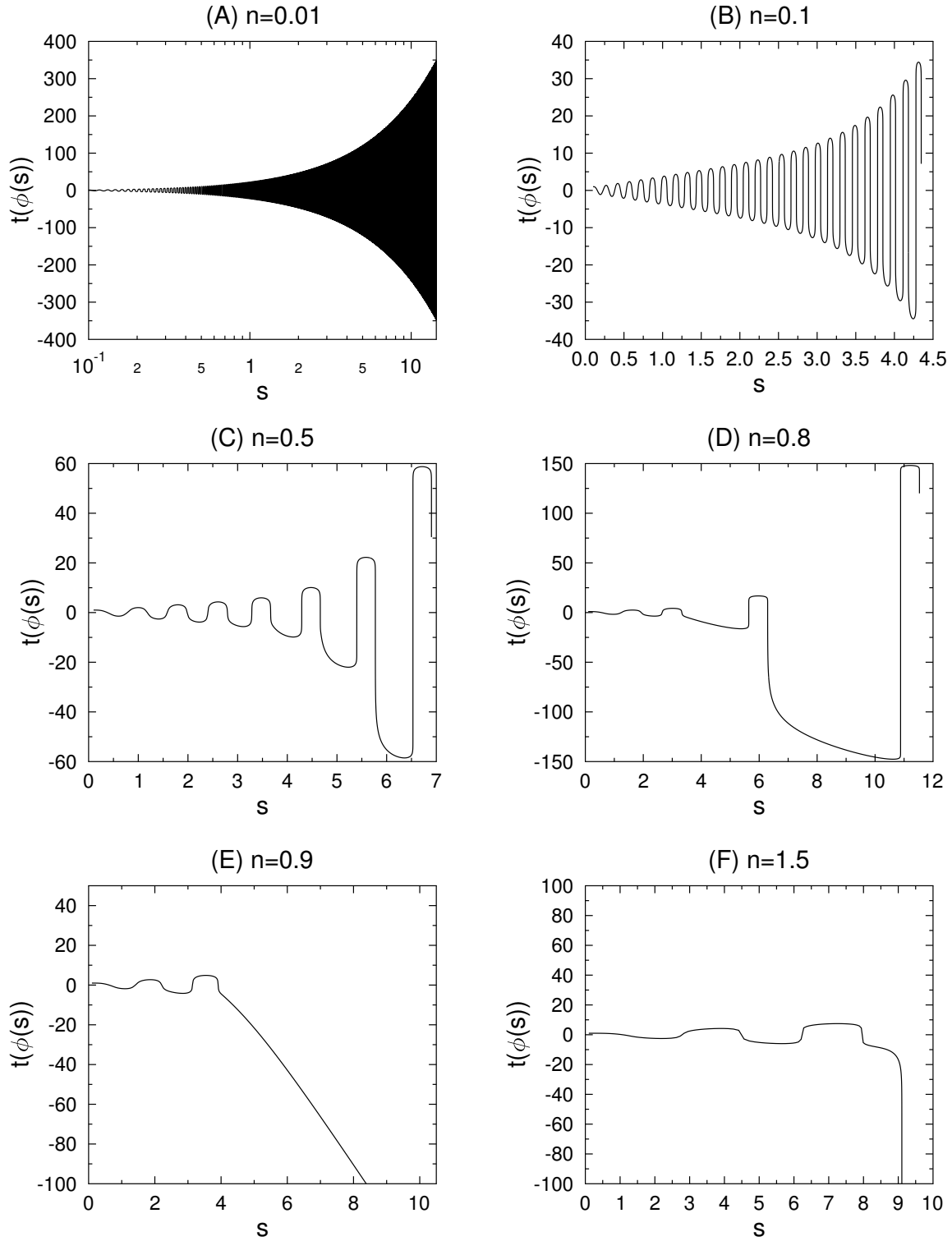


FIGURE 3. Illustrative numerical profiles of transformed  $\phi(s)$  via (6.3). Shown is the parameter case  $N = 1, \alpha = 0.5$  for selected  $n$ .

fig3

$N=1, \alpha=1$

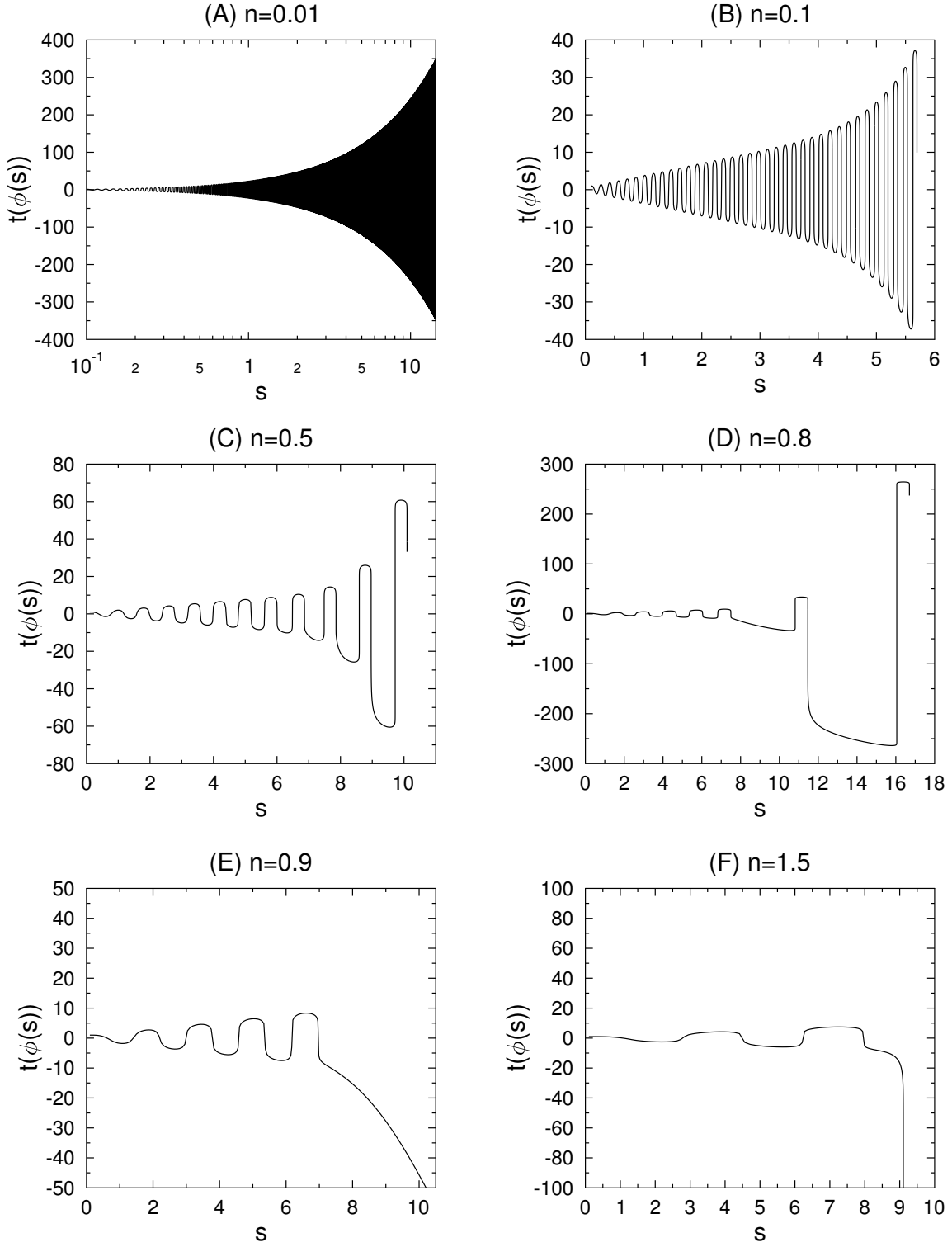


FIGURE 4. Illustrative numerical profiles of transformed  $\phi(s)$  via (6.3). Shown is the parameter case  $N = 1, \alpha = 1$  for selected  $n$ .

fig4

## 7. NONLINEAR EIGENFUNCTIONS BY SHOOTING

The eigenfunctions for  $n > 0$  may be obtained via shooting. The conditions (4.12) or (4.13) are again used as initial conditions and  $\alpha$  determined by capturing the growth (3.8) for sufficiently large  $y$  values. This growth behaviour may be imposed by minimising  $\mu y f' - \alpha f$  at  $y$  values typically chosen to be around 40. Standard regularisation of the term  $|f(y)|^n$  is required in the form  $(f^2 + \delta^2)^{\frac{n}{2}}$  with  $\delta$  taken relatively small. Figure 5 shows the eigenvalues of the first three branches  $k = 1, 2, 3$ . These numerically extend the  $n = 0$  eigenvalues of Proposition 4.1 to  $n > 0$ . As in the  $n = 0$  linear case, the eigenvalues remain the same irrespective of the spatial dimension  $N$ , at least to the accuracy of the numerical calculations that were performed. Furthermore, it is worth remarking that along each branch i.e. fixed  $k$ , the exponent  $\mu$  of the far-field behaviour of the eigenfunctions  $f \sim C y^\mu$  remains quite close to its value in the linear case i.e.  $\mu(n) \approx \mu(0)$ . Thus, as an approximation we have  $\alpha_k(n) \approx \alpha_k(0)/(1 - \alpha_k(0)n)$ , which seems to be a reasonable approximation at least for  $n < 1/\alpha_k(0)$ . The corresponding eigenfunctions are thus similar to those in the linear case.

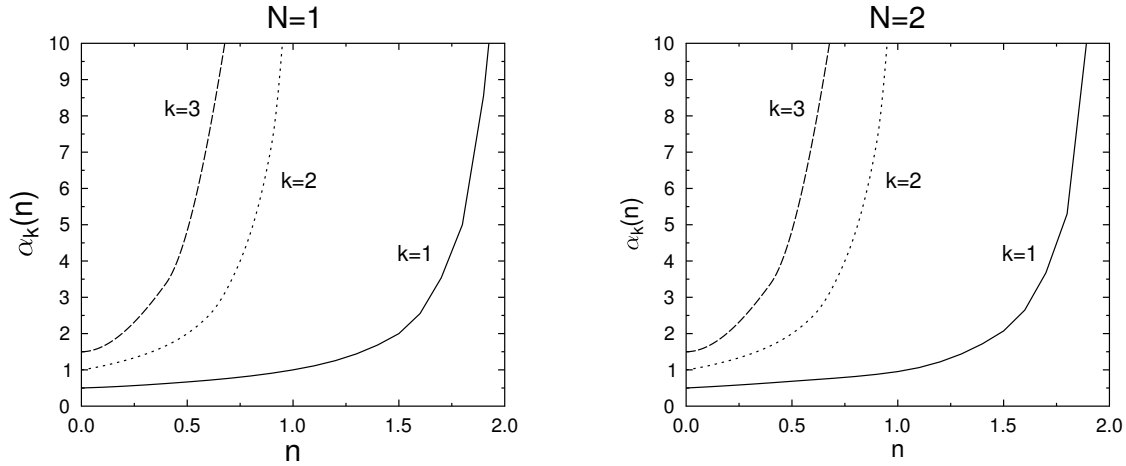


FIGURE 5. Numerical determination of the eigenvalues  $\alpha_k(n)$  for the first three branches  $k = 1, 2, 3$  of eigenfunctions.

fig5

An alternative approach to determining the eigenfunctions in both the linear  $n = 0$  and nonlinear  $n > 0$  cases, is through minimisation of the oscillatory maximal profile to obtain the non-oscillatory minimal profile for  $y \gg 1$ . In principle we have two parameters  $\alpha$  and say  $\nu = f''(0)$  or  $f(0)$  depending on which branch of eigenfunctions is considered. In the linear case, using (4.7), we are required to satisfy two algebraic equations with analytic functions:

$$\boxed{\text{sh3}} \quad (7.1) \quad \begin{cases} C_1(\alpha, \nu) = 0, \\ C_2(\alpha, \nu) = 0. \end{cases}$$

Therefore, we arrive at a well-posed 2 – 2 shooting problem, which cannot have more than a countable pairs of solutions (as mentioned already). This approach is practical for the linear  $n = 0$  case, as the two parameter form of the maximal bundle is known explicitly as given in (4.7) and was essentially pursued in section 4.2. However, lacking such an explicit expression for  $n > 0$ , limits this approach in the nonlinear case.

Finally it is worth presenting some numerical experiments showing the behaviour of the maximal profiles. Figure 6 shows profiles in the case  $N = 1, \alpha = 0.75$  for selected  $n$ . Since the oscillatory profiles occur over such large ranges, the transformed profile using (6.3) is depicted. The figures suggest the loss of a pure oscillatory structure as  $n$  increases, which is compatible with the behaviour seen for  $\phi$  in Figures 3 and 4. We conjecture that this is highly suggestive of a homoclinic-heteroclinic bifurcation. The precise values of  $n$  though at which this occurs is difficult to determine with sufficient accuracy.

## 8. AFTER-FOCUSING SELF-SIMILAR EXTENSION

As usual and as we have observed in many similar problems on a (unique) extension of a solution after blow-up, this is much easier. In fact, after focusing at  $t = 0^-$ , we arrive at a well-posed CP (or TFE, that does not matter since  $u_0(r) > 0$ , no interfaces are available) for the TFE-4 (1.1) with initial data (3.9) (with  $\mu = \mu_k$  when possible), already satisfying the necessary minimal growth at infinity.

Therefore, there exists a self-similar solution  $(2.1)_+$  with  $f(y)$  satisfying the ODE  $(2.3)_+$ , *with already fixed “eigenvalue”*  $\alpha = \alpha_k$ . Because of changing the signs in front of the linear terms, this changes the dimension of the stable/unstable manifolds as  $y \rightarrow +\infty$  (in particular, the unstable manifold becomes 1D, similar to  $n = 0$ ).

To explain the latter, as usual consider the case  $n = 0$  (and hence small  $n > 0$ ). Then, with the change of sign in the two linear terms in (4.3), we arrive at a different characteristic equation for  $a$ :

$$\boxed{\text{b3N}} \quad (8.1) \quad f(y) \sim e^{ay^\gamma} \quad \text{as } y \rightarrow +\infty \quad \implies \quad a^3 = +\frac{1}{4} \left(\frac{3}{4}\right)^3.$$

This characteristic equation gives two stable roots with  $\text{Re}(\cdot) < 0$ :

$$\boxed{\text{b4N}} \quad (8.2) \quad b_{1,2} = \frac{3}{4} 4^{-\frac{1}{3}} \left[ -\frac{1}{2} \pm i \frac{\sqrt{3}}{2} \right] \equiv c_0 \pm i c_1.$$

and one positive root maximal solutions; see below)

$$b_3 = \frac{3}{4} 4^{-\frac{1}{3}} > 0.$$

Hence, the bundle of maximal solutions is oscillatory as  $y \rightarrow +\infty$  is just 1D, so we arrive at an *undetermined* problem for  $\nu, \alpha$ , which satisfy just a single algebraic equation (unlike two in (7.1)). Indeed, this makes the  $\alpha$ -spectrum continuous and solvability for any  $\alpha > 0$ .

This allows us to get a unique extension profile  $f(y)$  for such a *a priori* fixed eigenvalue  $\alpha_k$ . In other words, the focusing extension problem is not an eigenvalue one, since a proper  $\alpha_k$  has been fixed by the previous focusing blow-up evolution.

It might be said, using standard terminology from linear operator theory, that for the sign + the spectrum of this problem becomes continuous, unlike the discrete one for  $-$ . Therefore, we do not study this much easier problem anymore, especially, since it has nothing to do with our goal: to detect an non-improvable regularity for the TFE-4 via blow-up focusing.

$N=1, \alpha=0.75$

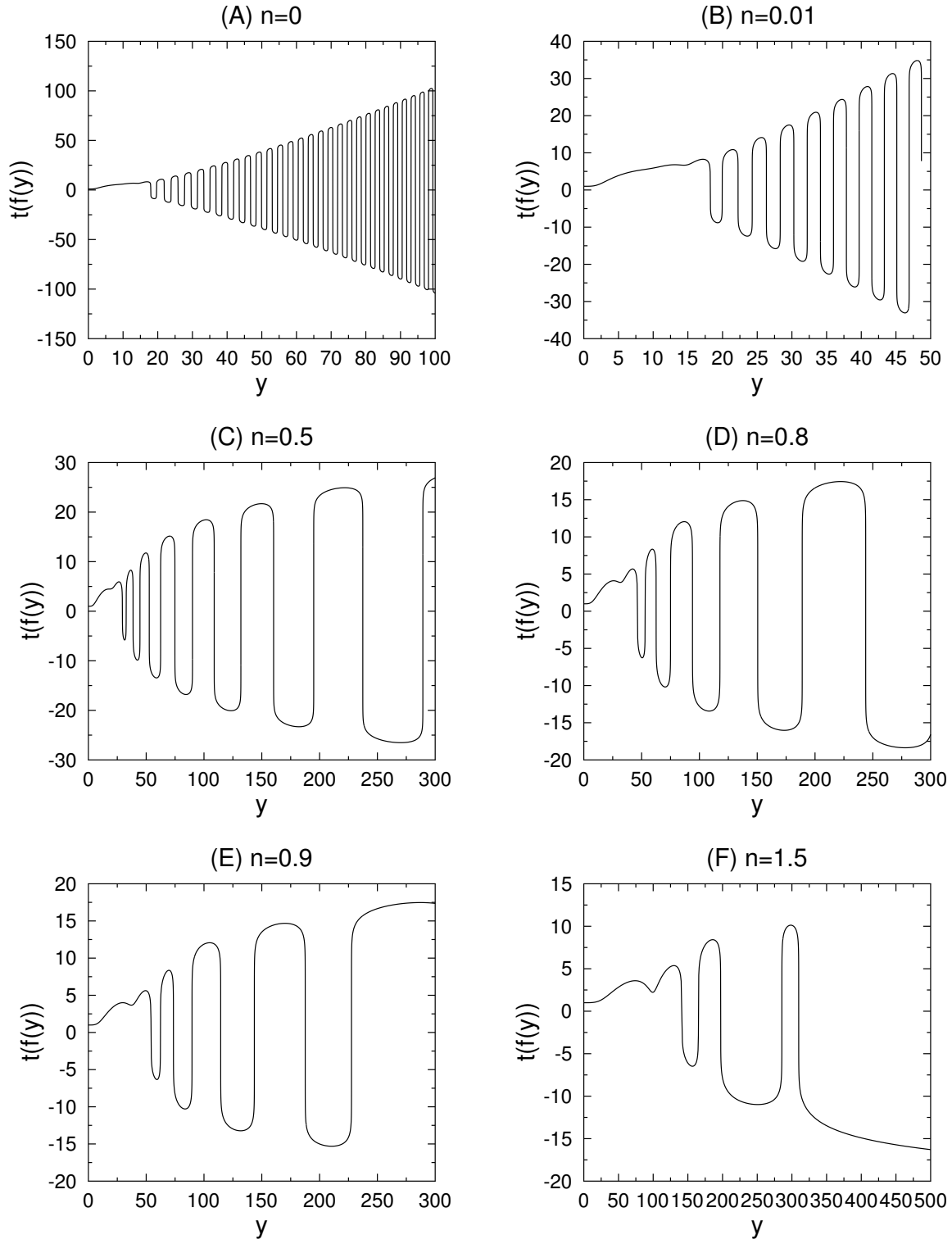


FIGURE 6. Illustrative numerical profiles of transformed  $\phi(s)$  via (6.3). Shown is the parameter case  $N = 1, \alpha = 1$  for selected  $n$ .

fig6

To ALL: is it OK above? To JDE: or solve it numerically, shooting is easier here, just a couple of examples, if really simple: shooting is the same; just change the signs...

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## APPENDIX A. THE LIMIT $n \rightarrow 0$ FOR MAXIMAL SOLUTIONS

We consider here the singular limit  $n \rightarrow 0^+$  for the equation in (3.7), written explicitly here as

$$\frac{1}{y^{N-1}} \left[ y^{N-1} |f|^n \left( \frac{1}{y^{N-1}} (y^{N-1} f')' \right)' \right]' + \frac{1}{4} (1 + \alpha n) y f' - \alpha f = 0.$$

21

The non-uniform solution in this limit comprises two regions, an *Inner region* where  $1 \ll y < O(n^{-3/4})$  with  $|f|^n \sim 1$ , and an *Outer region*  $y = O(n^{-3/4})$  where  $\ln |f| = O(1/n)$ . The labeling of these regions as inner and outer becomes apparent during the course of the scalings.

In the inner region  $1 \ll y < O(n^{-3/4})$ , we obtain at leading order in  $n$  the linear ODE (4.3). Posing  $f \sim f_0$  with  $|f_0|^n \sim 1$ , we obtain

$$\boxed{\text{inner1}} \quad (\text{A.1}) \quad \frac{1}{y^{N-1}} \left[ y^{N-1} \left( \frac{1}{y^{N-1}} (y^{N-1} f_0')' \right)' \right] + \frac{1}{4} (1 + \alpha n) y f_0' - \alpha f_0 = 0.$$

The far-field behaviour of (A.1) may be determined using a WKBJ expansion in the form

$$\boxed{\text{inner2}} \quad (\text{A.2}) \quad f_0(y) \sim A(y) e^{\phi(y)} \quad \text{as } y \rightarrow +\infty$$

which gives

$$\boxed{\text{inner3}} \quad (\text{A.3}) \quad (\phi')^3 = \frac{1}{4} y, \quad 3y A' + 2(N + 2\alpha - 1 - 12(\phi')^2 \phi'') A = 0.$$

The required solutions to (A.3) take the form

$$\boxed{\text{inner4}} \quad (\text{A.4}) \quad \phi(y) = a y^{4/3}, \quad A(y) = k y^{-\frac{2}{3}(N+2\alpha)},$$

where  $a$  satisfies the cubic equation in (4.4) and  $k$  an arbitrary constant. Thus, the dominant behaviour for large  $y$  is

$$\boxed{\text{inner5}} \quad (\text{A.5}) \quad f_0(y) \sim y^{-\frac{2}{3}(N+2\alpha)} \left( k_1 \exp \{a_1 y^{\frac{4}{3}}\} + k_2 \exp \{a_2 y^{\frac{4}{3}}\} \right),$$

with  $a_{1,2}$  as given in (4.5) and  $k_{1,2}$  complex constants chosen so that the expression is real as stated in (4.7).

This inner solution breaks down when  $y = O(n^{-3/4})$ , where  $\ln |f_0| = O(1/n)$ . This suggests the consideration of an outer region with scaling  $Y = n^{-3/4} y$ . In  $Y = O(1)$ , we have

$$\boxed{\text{outer1}} \quad (\text{A.6}) \quad \frac{n^3}{Y^{N-1}} \frac{d}{dY} \left[ Y^{N-1} |f|^n \frac{d}{dY} \left( \frac{1}{Y^{N-1}} \frac{d}{dY} \left( Y^{N-1} \frac{df}{dY} \right) \right) \right] + \frac{1}{4} (1 + \alpha n) Y \frac{df}{dY} - \alpha f = 0.$$

Rather than posing a multiple-scales ansatz directly, it is more convenient to work in complex form and we may instead consider

$$\boxed{\text{outer2}} \quad (\text{A.7}) \quad f(Y) \sim e^{b(Y)/n} B(Y) \quad \text{as } n \rightarrow 0,$$

where  $b$  is complex in order to match with the inner solution expression (A.5). Thus, at  $O(1/n)$  in (A.6) we obtain

$$\boxed{\text{outer3}} \quad (\text{A.8}) \quad |e^b| (b')^3 + \frac{1}{4} Y = 0,$$

whilst at  $O(1)$  we have

$$\boxed{\text{outer4}} \quad (\text{A.9}) \quad 3 \frac{B'}{B} + \left( 1 - \frac{\alpha}{4} + \ln |B| \right) b' + 6 \frac{b''}{b'} + \frac{2((N-1) + 2\alpha)}{Y} = 0,$$

where  $'$  denotes  $\frac{d}{dY}$  and the approximation

$$\boxed{\text{outer5}} \quad (\text{A.10}) \quad |f|^n \sim |e^b| (1 + n \ln |B|)$$

has been used. The solution to (A.8) that matches with (A.2) and (A.4) is

$$\boxed{\text{outer6}} \quad (\text{A.11}) \quad b(Y) = a \frac{3}{c_0} \ln \left( 1 + \frac{c_0}{3} Y^{4/3} \right).$$

The amplitude  $B(Y)$  is determined from (A.9) using the solution (A.11). We obtain

$$\ln |B| = \frac{1}{\left(1 + \frac{c_0}{3} Y^{4/3}\right)} \left( |k| - \frac{2}{3}(N + 2\alpha) \ln Y - \frac{c_0}{12}(2N + 3\alpha - 4) Y^{4/3} \right),$$

after matching to (A.2) with (A.4).

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